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Abstract

We compute (algebraically) the Euler characteristic of a complex of sheaves with constructible cohomology. A stratified Poincaré-Hopf formula is then a consequence of the smooth Poincaré-Hopf theorem and of additivity of the Euler-Poincaré characteristic with compact supports, once we have a suitable definition of index.

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1 Introduction

M.-H. Schwartz has defined radial vector fields in [Sch65a] and extended the classical Poincaré-Hopf theorem to real analytic sets, equipped with a Whitney stratification for these vector fields [Sch86], [Sch91]. In their turn, H. King and D. Trotman have extended M.-H. Schwartz's result to more general singular spaces and generic vector fields [KT06].

Radial [Sch65a], [Sch65b] (and totally radial, see [KT06], [Sim95]) vector fields are important because of their relation with Chern-Schwartz-Mac Pherson classes. Chern-Mac Pherson classes are written as an integral combination of Mather classes of algebraic varieties with coefficients determined by local Euler obstructions [Mac74]. A transcendental definition (and the original one) of local Euler obstruction is the obstruction to extend a lift of a radial vector field, prescribed on the link of a point in the base, inside a whole neighborood of Nash transform. Chern-Schwartz classes [Sch65a], [Sch65b] (which lie in cohomology of the complex analytic variety) are defined as the obstruction to extend a radial frame field given on a sub-skeleton of a fixed triangulation. These two points of view coincide: Chern-Mac Pherson classes are identified with Chern-Schwartz classes by Alexander duality [BS81]. In [BBF$^+$95], it is shown that these Chern-MacPherson-Schwartz classes can be realised (in general not uniquely) in intersection homology with middle perversity.

This paper concerns a Poincaré-Hopf theorem in intersection homology for a stratified pseudo-manifold A ([GM83]) and a vector field v which does not necessarily admit a globally continuous flow. Our main result is that we still have a Poincaré-Hopf formula when the vector field is semi-radial [KT06] :

$$I\chi_c^{\overline{p}}(A) = \sum_{v(x)=0} Ind^{\overline{p}}(v,x).$$

More precisely, we compute (algebraically) the Euler characteristic of a complex of sheaves with constructible cohomology. A stratified Poincaré-Hopf formula is then a consequence of the smooth Poincaré-Hopf theorem and of additivity of the Euler-Poincaré characteristic with compact supports, once we have a suitable definition of index.

Given a vector field with isolated singularities on a singular space, which admits a globally continuous flow, one can already deduce a Poincaré-Hopf theorem from a Lefschetz formula in intersection homology with middle perversity [GM85], [GM93], [Mac84].

A. Dubson announced in [Dub84] a formula similar to ours for a constructible complex in a complex analytic framework. In [BDK81], J.-L. Brylinski, A. Dubson and M. Kashiwara expressed the "local characteristic" of a holonomic module as a function of multiplicities of polar varieties and local Euler obstructions.

M. Goresky and R. MacPherson have proved a Lefschetz fixed point theorem for a subanalytic morphism and constructible complex of sheaves [GM93]. They show that a weakly hyperbolic morphism (*i.e.* whose fixed points are *weakly hyperbolic*) can be lifted to a morphism (not necessarily unique) at the level of sheaves. The Lefschetz number can be written as a sum of contributions of the various connected components of fixed points, a component being itself possibly stratified; every contribution is a sum of multiplicities (relative to the morphism), weighted by Euler characteristics in compactly supported cohomology of the strata of the connected component.

In Section 2 we give a formula to calculate the characteristic of a constructible complex of sheaves. Then, in section 3, we apply the preceding results to the intersection chain complex. A brief recall of definitions and results on stratified vector fields is given in section 4. A theorem of Poincaré-Hopf type appears in section 5, where the vector field considered is totally (or only semi-) radial. Sections 6 and 7 are devoted to illustrate the theorems of section 4.

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2 A formula to calculate the Euler-Poincaré characteristic of a complex of sheaves with constructible cohomology

First we recall some definitions. Let R be a principal ideal domain. We shall consider sheaves of $R-$modules.

Definition 2.1 *A* stratified set *A is a topological space which is a union of a locally finite family of disjoint, connected subsets (strata) which are smooth manifolds, satisfying the frontier condition. We shall denote by $\mathcal{A}$ the set of strata and suppose that this stratification is fixed once and for all.*

Definition 2.2 *Let A be a stratified set. We say that A is* compactifiable *if there exists a compact abstract stratified set $(B, \mathcal{B})$* ([Mat70], [Mat73], [Tho69], [Ver84])*, such that $A \subseteq B$ is a locally closed subset of B which is a union of elements of $\mathcal{B}$. We then say that $(B, \mathcal{B})$ is a* compactification *of A.*

Definition 2.3 *Let A be a stratified set and $\mathcal{F}$ a sheaf on A. We say that $\mathcal{F}$ is $\mathcal{A}-$*constructible *on A if for every stratum X of $\mathcal{A}$, the sheaf $\mathcal{F}_{|X}$ is locally constant of finite rank on R.*

Recall that $H_c(A; \mathcal{F}) \cong \mathbb{H}_c(A; \mathcal{F})$ where $\mathbb{H}_c$ denotes hypercohomology with compact supports. As usual, suppose that $H_c^p(A; \mathcal{F})$ has finite rank for $p \geq 0$ and is null for large enough p. Then we call *Euler characteristic of A with compact supports and coefficients in $\mathcal{F}$*, the alternating sum of the ranks of the modules $H_c^p(A; \mathcal{F})$ and denote it by $\chi^c(A; \mathcal{F})$. When the sheaf $\mathcal{F}$ is the constant sheaf R, we simply write $\chi^c(A)$. We shall see that the Euler characteristic is always defined in our situation.

Proposition 2.1 *Let X be a locally compact topological space, $\mathcal{G}$ a locally constant sheaf on X of finite rank g and suppose that X admits a finite partition $\mathcal{T}$ into open simplexes, i.e.*

there exists a finite simplicial complex (resp. subcomplex, possibly empty) K (resp. L) and a homeomorphism $\varphi : K \setminus L \to X$. Then

$$\chi^c(X;\mathcal{G}) = \chi^c(X).g.$$

Proof. As simplexes of X are contractible, the restriction of $\mathcal{G}$ is isomorphic to the constant sheaf over any one of them. Consider the finite union U of open simplexes of maximal dimension m. By induction on m and using the long cohomological exact sequence (with compact supports) of (U, X), we are reduced to showing the result for U. But, applying Mayer-Vietoris to the partition of U, this shows that $\chi^c(U;\mathcal{G}_{|U})$ is well defined and establishes the formula.

Proposition 2.2 *Let A be a compactifiable stratified set and $(B,\mathcal{B})$ a compactification of A. Let $\mathcal{A} = (X_i)_{i\in\{1,\dots,N\}}$ be the strata of A and $\mathcal{F}$ an $\mathcal{A}-$constructible sheaf. Then we have :*

$$\chi^c(A;\mathcal{F}) = \sum_{i=1}^{N} \chi^c(X_i).rk\ \mathcal{F}_{|X_i}.$$

Proof. Write $\overline{A}$ for the closure of A in B. Thanks to the triangulation theorem for abstract stratified sets of M. Goresky [Gor78], there exists a triangulation $\mathcal{T}$ of $\overline{A}$ adapted to the stratification $\overline{\mathcal{A}}$. As $\overline{A}$ is compact, this triangulation is finite. Moreover, it is also adapted to $\mathcal{A}$.
We are going to do induction on the number of strata of A and apply the method of proof of proposition 2.1. Let X be a stratum of maximal depth ([Ver84]) in A. Remark that X is closed in A. If $A = X$ we apply proposition 2.1 with X and $\mathcal{F}$.

Suppose the cardinal of $\mathcal{A}$ is strictly greater than 1.
We have then a long exact sequence in cohomology :

$$\cdots \to H^p_c(A\setminus X;\mathcal{F}_{|A\setminus X}) \to H^p_c(A;\mathcal{F}) \to H^p_c(X;\mathcal{F}_{|X}) \to \cdots$$

As the number of strata of $A \setminus X$ is strictly smaller than that in A, we can apply the induction hypothesis to $A \setminus X$ and $\mathcal{F}_{|A\setminus X}$. This shows that $rk\ H^p_c(A;\mathcal{F})$ is finite, so $\chi^c(A;\mathcal{F})$ is defined. On the other hand, we have :

$$\chi^c(A;\mathcal{F}) = \chi^c(A\setminus X;\mathcal{F}_{|A\setminus X}) + \chi^c(X;\mathcal{F}_{|X}).$$

We conclude by using the induction hypothesis and proposition 2.1.

Let $\mathcal{F}^\bullet$ be a complex of sheaves. Let $\mathcal{H}^\bullet(\mathcal{F}^\bullet)$ be the complex of derived sheaves.

Definition 2.4 *Let A be a compactifiable stratified set and $\mathcal{F}^\bullet$ a complex of sheaves on A. We say that $\mathcal{F}^\bullet$ has $\mathcal{A}-$*constructible cohomology *if :*

(i) $\mathcal{F}^\bullet$ is bounded

(ii) $\mathcal{H}^\bullet(\mathcal{F}^\bullet)$ is $\mathcal{A}-$constructible.

Theorem 2.1 *Let A be a compactifiable stratified set, $\mathcal{A} = (X_i)_{i\in\{1,\dots,N\}}$ its stratification and $\mathcal{F}^\bullet$ a complex of $c-$acyclic sheaves with $\mathcal{A}-$constructible cohomology. Then we have :*

$$\chi^c(A;\mathcal{F}^\bullet) = \sum_{q=-N_1}^{N_2} (-1)^q rk\ \mathbb{H}^q_c(A;\mathcal{F}^\bullet) = \sum_{i=1}^{N} \chi^c(X_i)\chi((\mathcal{H}^\bullet(\mathcal{F}^\bullet)_{|X_i})_{x_i})$$

where $\mathcal{F}^p = 0$ except for $-N_1 \le p \le N_2$ and x_i is any point of X_i, $1 \le i \le N$.

Proof. As $\mathcal{F}^p$ is $c-$acyclic for all $p \in \mathbb{Z}$, we have $H^p(H_c^q(A;\mathcal{F}^\bullet)) = 0$ for all $p \in \mathbb{Z}$ and $q \geq 1$. So the second spectral sequence, of second term ‘$E_2^{pq} = H^p(H_c^q(A;\mathcal{F}^\bullet))$, degenerates. As $\mathcal{F}^\bullet$ is bounded, the filtration of the associated double complex is regular, so the first spectral sequence is convergent and we have according to theorem 4.6.1 of [God73] p. 178 :

$$E_2^{p,q} = H_c^p(A;\mathcal{H}^q(\mathcal{F}^\bullet)) \Rightarrow \mathbb{H}_c^{p+q}(A;\mathcal{F}^\bullet).$$

As $\mathcal{F}^\bullet$ has $\mathcal{A}-$constructible cohomology and A is compactifiable, we can define :

$$\begin{aligned}
\chi(E_2) &= \textstyle\sum_{p\in\mathbb{N},q\in\mathbb{Z}}(-1)^{p+q} rg\ H_c^p(A;\mathcal{H}^q(\mathcal{F}^\bullet)) \\
&= \textstyle\sum_{q\in\mathbb{Z}}(-1)^q\sum_{p\in\mathbb{N}}(-1)^p rg\ H_c^p(A;\mathcal{H}^q(\mathcal{F}^\bullet)) \\
&= \textstyle\sum_{q=-N_1}^{N_2}(-1)^q\chi^c(A;\mathcal{H}^q(\mathcal{F}^\bullet))
\end{aligned}$$

for $\mathcal{F}^\bullet$ is bounded. Remark that, since A is triangulable, every point of A (which is paracompact) admits a neighborhood homeomorphic to a subspace of some $\mathbb{R}^p$, so that A is of cohomological dimension lower or equal to p ($< \infty$ because A is compactifiable), according to theorem 5.13.1 of [God73] p. 237.

Apply then proposition 2.2 to A and $\mathcal{H}^q(\mathcal{F}^\bullet)$:

$$\begin{aligned}
\chi(E_2) &= \textstyle\sum_{q=-N_1}^{N_2}(-1)^q\sum_{i=1}^{N}\chi^c(X_i) rg\ (\mathcal{H}^q(\mathcal{F}^\bullet)_{|X_i})_{x_i} \\
&= \textstyle\sum_{i=1}^{N}\chi^c(X_i)\sum_{q=-N_1}^{N_2}(-1)^q rg\ (\mathcal{H}^q(\mathcal{F}^\bullet)_{|X_i})_{x_i} \\
&= \textstyle\sum_{i=1}^{N}\chi^c(X_i)\sum_{q=-N_1}^{N_2}(-1)^q rg\ \mathcal{H}^q(\mathcal{F}^\bullet)_{x_i} \\
&= \textstyle\sum_{i=1}^{N}\chi^c(X_i)\chi(\mathcal{H}^\bullet(\mathcal{F}^\bullet)_{x_i})
\end{aligned}$$

with $x_i \in X_i$ for $i \in \{1,\ldots,N\}$. As $E_{r+1} = H(E_r)$, we have $\chi(E_{r+1}) = \chi(E_r)$ for all $r \geq 2$. So $\chi(E_r) = \chi(E_2)$ for all $r \geq 2$.
As $\mathcal{F}^\bullet$ is bounded, $E_2^{p,q} = 0$ for q big enough or small enough and $p \in \mathbb{N}$. Thus the spectral sequence degenerates and so

$$E_r^{p,q} = E_\infty^{p,q}$$

for r big enough.
Hence

$$\chi(E_\infty) = \chi(E_r) = \chi(E_2).$$

But $(E_\infty^{p,q})_{p+q=s}$ is the associated graded module to $\mathbb{H}_c^s(A;\mathcal{F}^\bullet)$. We have thus :

$$rg\ \mathbb{H}_c^s(A;\mathcal{F}^\bullet) = \sum_{p+q=s} rg\ E_\infty^{p,q}.$$

Finally

$$\begin{aligned}
\chi^c(A;\mathcal{F}^\bullet) &= \textstyle\sum_{s\in\mathbb{Z}}(-1)^s rg\ \mathbb{H}_c^s(A;\mathcal{F}^\bullet) \\
&= \chi(E_\infty) \\
&= \chi(E_2) \\
&= \textstyle\sum_{i=1}^{N}\chi^c(X_i)\chi(\mathcal{H}^\bullet(\mathcal{F}^\bullet)_{x_i}).
\end{aligned}$$

Remark. Theorem 2.1 works also with the weaker hypothesis of (finite) triangulabity.

3 Application to intersection homology

Suppose now that A is a pseudo-manifold, and let $\mathcal{A}$ be its stratification. Here the strata of A will no longer be necessarily connected, but we shall work with connected components of strata. We denote by L_x the link of the point x in A.

Proposition 3.1 ([Ba84]) *Let A be an n-pseudo-manifold and $\overline{p}$ a perversity. Let $IC^{\overline{p}}_{\bullet}$ be the intersection chain complex for perversity $\overline{p}$ with coefficients in R* [GM93] *and set $\mathcal{IC}^{\bullet}_{\overline{p}}$ =sheaf associated to the presheaf $\{U \mapsto IC^{\overline{p}}_{n-\bullet}(U)\}$. Then $\mathcal{IC}^{\bullet}_{\overline{p}}$ is a complex of $c-$soft sheaves (so $c-$acyclic). Moreover we have :*

$$\mathbb{H}^{\bullet}_{c}(A; \mathcal{IC}^{\bullet}_{\overline{p}}) = IH^{\overline{p}}_{n-\bullet}(A; R).$$

Proposition 3.2 (Proposition 2.4 of [GM83]) *Let A be an n-pseudo-manifold, x any point in a stratum X^k of A of dimension k and L_x the link of X^k at x in A. The fibre of the complex of derived sheaves $\mathcal{H}^{\bullet}(\mathcal{IC}^{\bullet}_{\overline{p}})$ is given by :*

$$\mathcal{H}^i(\mathcal{IC}^{\bullet}_{\overline{p}})_x = \begin{cases} \begin{cases} IH^{\overline{p}}_{n-i-k-1}(L_x) & \text{if } i \le p_{n-k} \\ 0 & \text{otherwise} \end{cases} & \text{if } x \in X^k \subset A \setminus A_{reg} \\ \begin{cases} R & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} & \text{if } x \in X^n \subset A_{reg}. \end{cases}$$

As usual the *Euler-Poincaré characteristic in intersection homology* $I\chi^{\overline{p}}_c(A)$ of an n-pseudo-manifold A is the Euler-Poincaré characteristic with compact supports of the complex of sheaves $\mathcal{IC}^{\bullet}_{\overline{p}}$ multiplied by $(-1)^n$, *i.e.* $I\chi^{\overline{p}}_c(A) = (-1)^n \chi^c(A; \mathcal{IC}^{\bullet}_{\overline{p}})$.

Theorem 3.1 *Let A be an n-pseudo-manifold such that $(A, \mathcal{A})$ is compactifiable, N the number of connected components of strata of A and $\overline{p}$ a perversity.*
We have :

$$I\chi^{\overline{p}}_c(A) = \sum_{i=1}^{N} (-1)^n \chi^c(X_i) \sum_{j=0}^{p_{n-\dim X_i}} (-1)^j rg\ IH^{\overline{p}}_{n-j-\dim X_i - 1}(L_{x_i}; R)$$

where x_i is an arbitrary point of X_i for $1 \le i \le N$ and we make the convention that $rg\ IH^{\overline{p}}_{-1}(L_{x_i}; R) = 1$ if $\dim X_i = n$.

Proof. Application of theorem 2.1 and proposition 3.2.

Remark. As in theorem 2.1 we can weaken the hypothesis by only assuming the existence of a (finite) triangulation compatible with the stratification.

Proposition 3.3 *Let A be a $2n$-pseudo-manifold such that $(A, \mathcal{A})$ is compactifiable, the dimension of strata being even and let $\overline{m}$ be the middle perversity. We have :*

$$I\chi^{\overline{m}}_c(A) = \sum_{i=0}^{n} \sum_{k=1}^{N_i} \chi_c(X^{2i}_k) \sum_{j=0}^{n-i-1} (-1)^j rg\ IH^{\overline{m}}_{2n-j-2i-1}(L_{x^i_k}; R)$$

where we have written X^{2i}_k (resp. N_i) for the k-th connected component of the stratum (resp. number of connected components of the stratum) of dimension $2i$, x^k_i an arbitrary point of X^{2i}_k and $\chi_c(X) = \sum_{i=0}^{\dim X} (-1)^i rg\ H^c_i(X; R)$.

Proof. We apply theorem 2.1 with $\overline{p} = \overline{m}$ and we remark that $\chi_c(X) = \chi^c(X)$ for a manifold X of even dimension.

4 Totally radial and semi-radial vector fields on abstract stratified sets

M.-H. Schwartz constructed certain frame fields to define (by obstruction) her Chern-Schwartz classes in the cohomology of a singular complex analytic variety equipped with a Whitney stratification [Sch65a], [Sch65b]. These were called radial fields. When one is concerned with 1-frame fields (i.e. vector fields), they are called radial vector fields. She showed that they verified a Poincaré-Hopf formula [Sch86], [Sch91].

This section is an easy transcription to abstract stratified sets of some notions and results of [KT06] which were given in the more general setting of "mapping cylinder stratified space with boundary". In their paper, H. King and D. Trotman extend M.-H. Schwartz's work on Poincaré-Hopf formulas, to more general spaces, and to generic vector fields. Notice that abstract stratified sets are not (necessarily) embedded nor are vector fields (necessarily) continuous.

Definition 4.1 ([KT06]) *Let $(A, \mathcal{A})$ be an abstract stratified set and v a stratified vector field on A (*[Mat70],[Mat73], [Tho69], [Ver84]*). We say that v is a* totally radial *vector field if for all strata $X \in \mathcal{A}$ there exists a neighborhood U_X of X in the control tube T_X such that $d\rho_X(v) > 0$ on $U_X \setminus X$ (i.e. v is pointing outwards with respect to the level hypersurfaces of the control function ρ_X).*

In [KT06] such a vector field was called *radial*. To avoid confusion with the radial vector fields of M.-H. Schwartz, we have adopted the terminology *totally radial*, which also expresses the fact that one imposes that $d\rho_X(v) > 0$ on a whole neighborhood U_X of X in T_X. The analoguous condition is only imposed on a neighbourhood of some closed subset of X by M.H. Schwartz. See [Sim95] for a detailed discussion of the differences between the radial fields of [Sch86], [Sch91] and the radial fields of [KT06], called totally radial here.

Proposition 4.1 *Let $(A, \mathcal{A})$ be an abstract stratified set and Y a stratum of A. Then there exists a vector field ξ_Y on $T_Y \setminus Y$ such that :*

$$\text{for all } y \text{ in } T_Y \setminus Y \text{ we have } \begin{cases} \rho_{Y*y}(\xi_Y) = 1 \\ \rho_{X*y}(\xi_Y) = 0 \quad \text{if } X < Y. \end{cases}$$

Proof. It suffices to consider the stratified submersion $(\pi_Y, \rho_Y) : T_Y \setminus Y \to Y \times \mathbb{R}_+^*$ and to lift the constant field $(0, \partial_t)$ to a field ξ_Y on $T_Y \setminus Y$. Thanks to the compatibility conditions, we see that $\rho_{X*}(\xi_Y) = 0$ for $X < Y$.

Definition 4.2 ([KT06]) *Let $(A, \mathcal{A})$ be an abstract stratified set, v a stratified vector field on A and Y a stratum of A. Let $(Y_i)_{1 \le i \le m}$ be the strata such that $Y < Y_i$. Set $B_Y(v) = \{x \in T_Y \setminus Y | (\exists c_i \in \mathbb{R}_- | 0 \le i \le m) : v(y) = c_0 \xi_Y(y) + \sum_{j=1}^m c_i \xi_{Y_i}(y)$ with $c_0 < 0\}$. A point $x \in \overline{B_Y}(v) \cap Y$ is called a* virtual zero *of v.*

Definition 4.3 ([KT06]) *Let $(A, \mathcal{A})$ be an abstract stratified set and v a stratified vector field on A. Then v is called* semi-radial *if v has no virtual zero.*

Examples. Totally radial vector fields, and controlled vector fields, are semi-radial.

Definition 4.4 *Let $(A, \mathcal{A})$ be a compactifiable stratified set, $(B, \mathcal{B})$ a compactification of A such that $\mathcal{A} \subseteq \mathcal{B}$ and v a stratified vector field on A. We say that v is* strongly totally radial *(resp.* strongly semi-radial*) if and only if there exists a totally radial (resp. semi-radial) extension u of v to $(B, \mathcal{B})$.*

Lemma 4.1 ([KT06]) *Let $(A, \mathcal{A})$ be an abstract stratified set (resp. compactifiable stratified set) and v a semi-radial (resp. strongly semi-radial) vector field with isolated singularities on A. Then there exists a (resp. strongly) totally radial vector field v' having the same singularities as v and the same indices at these points.*

5 Towards a Poincaré-Hopf theorem

Definition 5.1 *Let A be an n-pseudo-manifold, $\overline{p}$ a perversity and x a point of a stratum X. We call* multiplicity *of A at x for perversity $\overline{p}$ the following integer :*

$$m_x^{\overline{p}}(A) = \begin{cases} \sum_{i=n-p_{n-\dim X}}^{n} (-1)^i rg\ IH_{i-\dim X-1}^{\overline{p}}(L_x; R) & \text{if } x \in A \setminus A_{reg} \\ (-1)^n & \text{if } x \in A_{reg}. \end{cases}$$

Remark. The multiplicity is nothing else than $I\chi_c^{\overline{p}}(A, A - \{x\})$ (which equals $(-1)^n$ if $x \in A_{reg}$).

Definition 5.2 *Let A be an n-pseudo-manifold such that $(A, \mathcal{A})$ is a compactifiable abstract stratied set, $\overline{p}$ a perversity and v a stratified vector field having an isolated singularity at $x \in X$. We call* singular index *of v at x, and we denote by $Ind^{\overline{p}}(v, x)$ the integer:*

$$Ind^{\overline{p}}(v, x) = m_x^{\overline{p}}(A).Ind(v, x).$$

Recall that if the stratum X is reduced to a point, then $Ind(v, x) = 1$.

Theorem 5.1 *Let A be an n-pseudo-manifold such that $(A, \mathcal{A})$ is a compactifiable abstract stratified set, $\overline{p}$ a perversity and v a strongly semi-radial vector field admitting a finite number of singularities on A. We have :*

$$I\chi_c^{\overline{p}}(A) = \sum_{v(x)=0} Ind^{\overline{p}}(v, x).$$

Proof. As in [Bek92], for all strata X of A, let f_X be a carpeting function, i.e. let $U_{b(X)}$ be a neighborhood of $b(X) = \overline{X} \setminus X$ in $\overline{X}$, and let $f_X : U_{b(X)} \to \mathbb{R}_+$ be a continuous function (constructed using the control functions $\{\rho_X\}_{X \subseteq A}$ induced by the compactification of A), smooth on the stratum X such that $f_X^{-1}(0) = b(X)$ and $f_{X|U_{b(X)} \cap X}$ is submersive. Now, apply lemma 4.1 to v; this gives a totally radial vector field v'. Then we remark that if v' is a totally radial vector field, for all strata X, v' is entering $X_{\geq \epsilon} = X \setminus \{f_X < \epsilon\}$ along ∂X_ϵ for ϵ small enough, where the symbol ∂X_ϵ denotes the level hypersurface $\{f_X = \epsilon\}$. This is because $grad(f_X) = \sum_{Y<X} a_Y.grad(\rho_Y)$, where the a_Y are non-negative smooth functions, at every point of X. So we have $\chi_c(X_{\geq\epsilon}) - \chi_c(\partial X_\epsilon) = \sum_{v'(x)=0} Ind(v', x)$ thanks to the classical Poincaré-Hopf theorem. Finally, we have $\chi^c(M) = \chi_c(M) - \chi_c(\partial M)$ for every compactifiable manifold M by adding a boundary ∂M. Use the "additivity" formula of theorem 3.1 and the definition of the singular index to complete the proof.

6 A few examples

In the following computations, as we are only interested in the rank of intersection homology groups, we shall take $R = \mathbb{Q}$ and work with the dimension of $\mathbb{Q}-$vector spaces. Moreover, this will permit us to apply Poincaré duality to calculate some associated groups. In the remainder of the text, T^2 will denote the torus $S^1 \times S^1$. The stratifications of spaces will be the evident ones and we shall not go into details. See [Ba84] for classical tools to compute $IH_\bullet$ of the following spaces.

6.1 An inevitable example : the pinched torus T_p^2

We have a unique perversity $\overline{p} = \overline{0}$ and we have evidently a totally radial vector field u on T_p^2 with a unique singularity at the isolated singular point x_0 of T_p^2, of indice 1. The link at this point is $L_{x_0} = S^1 \sqcup S^1$. We have :

$$IH_i^{\overline{0}}(T_p^2) = \begin{cases} \mathbb{Q} & \text{si } i = 2 \\ 0 & \text{si } i = 1 \\ \mathbb{Q} & \text{si } i = 0 \end{cases}$$

so that

$$I\chi^{\overline{0}}(T_p^2) = 2.$$

On the other hand :

$$\begin{aligned} m_{x_0}^{\overline{0}}(T_p^2) &= \dim IH_1^{\overline{0}}(L_{x_0}) \\ &= \dim H_1(S^1 \sqcup S^1) \\ &= 2. \end{aligned}$$

Finally we have $I\chi^{\overline{0}}(T_p^2) = 2 = 2.1 = Ind^{\overline{0}}(u, x_0)$.

6.2 A well-known example : the suspension of the torus ΣT^2 (H. Poincaré, 1895)

This time, we have two different perversities $\overline{0}$ and $\overline{t}$ and two isolated singularities (which are the two vertices of suspension). The link at these points is $L_{x_0} = L_{x_1} = T^2$. We still have a totally radial vector field v with two singular points of indice 1 at singularities of ΣT^2. Remark that this pseudo-manifold is normal so we have $IH_*^{\overline{t}}(\Sigma T^2) = H_*(\Sigma T^2)$, i.e.

$$IH_i^{\overline{t}}(\Sigma T^2) = \begin{cases} \mathbb{Q} & \text{if } i = 3 \\ \mathbb{Q}^2 & \text{if } i = 2 \\ 0 & \text{if } i = 1 \\ \mathbb{Q} & \text{if } i = 0. \end{cases}$$

Hence

$$I\chi^{\overline{t}}(\Sigma T^2) = 2$$

and by duality we find

$$I\chi^{\overline{0}}(\Sigma T^2) = -2.$$

On the other hand :

$$\begin{aligned} m_{x_0}^{\overline{0}}(\Sigma T^2) &= -\dim IH_2^{\overline{0}}(L_{x_0}) \\ &= -\dim H_2(T^2) \\ &= -1 \end{aligned}$$

and

$$\begin{aligned} m_{x_0}^{\overline{t}}(\Sigma T^2) &= \dim IH_1^{\overline{t}}(L_{x_0}) - \dim IH_2^{\overline{t}}(L_{x_0}) \\ &= \dim H_1(T^2) - \dim H_2(T^2) \\ &= 1. \end{aligned}$$

Finally we have :

$$I\chi^{\overline{0}}(\Sigma T^2) = -2 = -1 - 1 = 2.Ind^{\overline{0}}(v, x_0)$$

and

$$I\chi^{\overline{t}}(\Sigma T^2) = 2 = 1 + 1 = 2.Ind^{\overline{t}}(v, x_0).$$

6.3 A hybrid example : the suspension of the torus of dimension 3, twice pinched, ΣT^3_{2p}

We have $\Sigma T^3_{2p} = \Sigma(\Sigma(T^2 \sqcup T^2))$. Here we have four perversities $\overline{0}, \overline{m}, \overline{n}, \overline{t}$. Calculate to begin with the homology of T^3_{2p} :

$$H_i(T^3_{2p}) = \begin{cases} \mathbb{Q}^{\,2} & \text{si } i = 3 \\ \mathbb{Q}^{\,4} & \text{if } i = 2 \\ \mathbb{Q} & \text{if } i = 1 \\ \mathbb{Q} & \text{if } i = 0 \end{cases}.$$

Then its intersection homology is :

$$\begin{aligned} IH^{\overline{0}}_i(T^3_{2p}) &= \begin{cases} H_i(T^3_{2p}) & \text{if } i > 2 \\ Im\ (H_i(T^2 \sqcup T^2) \to H_i(T^3_{2p})) & \text{if } i = 2 \\ H_i(T^2 \sqcup T^2) & \text{if } i < 2 \end{cases} \\ &= \begin{cases} \mathbb{Q}^{\,2} & \text{if } i = 3 \\ 0 & \text{if } i = 2 \\ \mathbb{Q}^{\,4} & \text{if } i = 1 \\ \mathbb{Q}^{\,2} & \text{if } i = 0 \end{cases} \end{aligned}$$

where we deduce

$$IH^{\overline{t}}_i(T^3_{2p}) = \begin{cases} \mathbb{Q}^{\,2} & \text{if } i = 3 \\ \mathbb{Q}^{\,4} & \text{if } i = 2 \\ 0 & \text{if } i = 1 \\ \mathbb{Q}^{\,2} & \text{if } i = 0 \end{cases}.$$

And at last the intersection homology of the suspension ΣT^3_{2p} is :

$$\begin{aligned} IH^{\overline{p}}_i(\Sigma T^3_{2p}) &= \begin{cases} IH^{\overline{p}}_{i-1}(T^3_{2p}) & \text{if } i > 3 - p_4 \\ 0 & \text{if } i = 3 - p_4 \\ IH^{\overline{p}}_i(T^3_{2p}) & \text{if } i < 3 - p_4 \end{cases} \\ &= \begin{cases} \begin{cases} \mathbb{Q}^{\,2} & \text{if } i = 4 \\ 0 & \text{if } i = 3 \\ 0 & \text{if } i = 2 \\ \mathbb{Q}^{\,4} & \text{if } i = 1 \\ \mathbb{Q}^{\,2} & \text{if } i = 0 \end{cases} & \text{if } \overline{p} = \overline{m} \\ \begin{cases} \mathbb{Q}^{\,2} & \text{if } i = 4 \\ 0 & \text{if } i = 3 \\ 0 & \text{if } i = 2 \\ \mathbb{Q}^{\,4} & \text{if } i = 1 \\ \mathbb{Q}^{\,2} & \text{if } i = 0 \end{cases} & \text{if } \overline{p} = \overline{0}. \end{cases} \end{aligned}$$

It is easy to construct a totally radial vector field w with four singularities : two at the vertices of suspension, say x_0, x_1, of indice 1 and two others on strata of codimension 3, say x_2, x_3, of indice -1. Links are $L_{x_0} = L_{x_1} = T^3_{2p}$ and $L_{x_2} = L_{x_3} = T^2 \sqcup T^2$. Calculations of multiplicities give :

$$m^{\overline{p}}_{x_0}(\Sigma T^3_{2p}) = \begin{cases} \dim IH^{\overline{t}}_1(T^3_{2p}) - \dim IH^{\overline{t}}_2(T^3_{2p}) + \dim IH^{\overline{t}}_3(T^3_{2p}) = -2 & \text{if } \overline{p} = \overline{t} \\ -\dim IH^{\overline{t}}_2(T^3_{2p}) + \dim IH^{\overline{t}}_3(T^3_{2p}) = -2 & \text{if } \overline{p} = \overline{n} \\ -\dim IH^{\overline{0}}_2(T^3_{2p}) + \dim IH^{\overline{0}}_3(T^3_{2p}) = 2 & \text{if } \overline{p} = \overline{m} \\ \dim IH^{\overline{0}}_3(T^3_{2p}) = 2 & \text{if } \overline{p} = \overline{0} \end{cases}$$

and

$$m^{\overline{p}}_{x_2}(\Sigma T^3_{2p}) = \begin{cases} -\dim H_1(T^2 \sqcup T^2) + \dim H_2(T^2 \sqcup T^2) = -2 & \text{if } \overline{p} = \overline{t} \\ -\dim H_1(T^2 \sqcup T^2) + \dim H_2(T^2 \sqcup T^2) = -2 & \text{if } \overline{p} = \overline{n} \\ \dim H_2(T^2 \sqcup T^2) = 2 & \text{if } \overline{p} = \overline{m} \\ \dim H_2(T^2 \sqcup T^2) = 2 & \text{if } \overline{p} = \overline{0} \end{cases}.$$

Finally we have

$$\begin{array}{ll} I\chi^{\overline{0}}(\Sigma T^3_{2p}) & = 0 = 2 + 2 + 2.(-1) + 2.(-1) \\ I\chi^{\overline{m}}(\Sigma T^3_{2p}) & = 0 = 2 + 2 - 2 - 2 \\ I\chi^{\overline{n}}(\Sigma T^3_{2p}) & = 0 = -2 - 2 + (-2).(-1) + (-2).(-1) \\ I\chi^{\overline{t}}(\Sigma T^3_{2p}) & = 0 = -2 - 2 + 2 + 2. \end{array}$$

7 A partial converse

We present here a partial converse to theorem 5.1 in the sense that we study when a stratified set admits a strongly totally radial vector field without singularity. This result is in the line of [Sul71], [Ver72] or [Sch91], [Sch92]. See also [Mat73], theorem 8.5. The result is partial because of the example below. Indeed, it shows that we cannot expect the condition $I\chi^{\overline{p}}_c(A) = 0$ to imply the existence of a non singular totally radial vector field.

Theorem 7.1 *Let A be a compactifiable n-pseudo-manifold. There exists a strongly totally radial vector field (relatively to $\mathcal{A}$) on A without singularity if and only if $\chi^c(X) = 0$ for all strata X of $\mathcal{A}$.*

Proof. To show sufficiency, we use the carpeting functions of the proof of theorem 5.1. Let v be a strongly totally radial vector field on A with isolated singularities ; the vector field v_X is entering on the boundary $\partial X_{\geq \epsilon}$ (defined by a level hypersurface of a carpeting function). Remark that, as $\chi^c(X) = 0$, we can deform v_X on $X_{\geq \epsilon}$ (without modifying it near $\partial X_{\geq \epsilon}$) so as to have no singularities ([Hir88]). We have evidently $\rho_{X*}(v) > 0$ on $T_X \setminus X$ for all strata X. Necessity is proved in an analogous manner.

Corollary 7.1 *Let A be a compactifiable n-pseudo-manifold, stratified with strata of odd dimension. Then there exists a strongly totally radial vector field without singularity on A.*

Remark. Existence of a totally radial vector field without singularity, on an abstract stratified set, is equivalent to the existence of a controlled vector field without singularity.

Example. Finally, here is an example of a compact pseudo-manifold without strata of dimension 0 for which $I\chi^{\overline{p}}_c(A) = 0$ for every perversity $\overline{p}$ and admitting no totally radial vector field without a singularity. Consider $A = \Sigma(T^3_{2p}) \times S^2$; it is clear that $I\chi^{\overline{p}}(A; R) = 0$ for all $\overline{p}$. Nevertheless, there does not exist a totally radial vector field without a singularity (look at strata $\{*\} \times S^2$ or $\{**\} \times S^2$). This is also evident as a consequence of theorem 7.1.

References

[Ba84] A. Borel and al. *Intersection cohomology*, volume 50 of *Progress in mathematics*. Birkhäuser Boston • Basel • Stuttgart, 1984.

[BBF+95] G. Barthel, J.-P. Brasselet, K.-H. Fieseler, O. Gabber, and L. Kaup. Relèvement de cycles algébriques et homomorphismes associés en homologie d'intersection. *Annals of Mathematics*, 141:147–179, 1995.

[BDK81] J.-L. Brylinski, A. Dubson, and M. Kashiwara. Formule d'indice pour les modules holonomes et obstruction d'Euler locale. *Comptes Rendus de l'Académie des Sciences*, 293:573–576, 1981.

[Bek92] K. Bekka. Continuous vector fields and Thom's isotopy theorems, 1992.

[BS81] J.-P. Brasselet and M.-H. Schwartz. Sur les classes de Chern d'un ensemble analytique complexe. In S.M.F., editor, *Séminaire E.N.S. 1978-1979 : Caractéristique d'Euler-Poincaré*, volume 82-83 of *Astérisque*, pages 93–147, 1981.

[Dub84] A. Dubson. Formule pour l'indice des complexes constructibles et des modules holonomes. *Comptes Rendus de l'Académie des Sciences*, 298(6):113–116, 1984.

[GM83] M. Goresky and R. MacPherson. Intersection homology theory II. *Inventiones Mathematicae*, 72:77–129, 1983.

[GM85] M. Goresky and R. MacPherson. Lefschetz fixed point theorem for intersection homology. *Commentarii Mathematici Helvetici*, 60(3):386–391, 1985.

[GM93] M. Goresky and R. MacPherson. Local contribution to the Lefschetz fixed point formula. *Inventiones Mathematicae*, 111:1–33, 1993.

[God73] R. Godement. *Topologie algébrique et théorie des faisceaux.* Actualités scientifiques et industrielles de l'Université de Strasbourg. Hermann Paris, 1973.

[Gor78] M. Goresky. Triangulation of stratified objects. *Proceedings of the American Mathematical Society*, 72:193–200, 1978.

[Hir88] M. Hirsch. *Differential topology*, volume 33. Springer-Verlag New York, 1988.

[KT06] H. King and D. Trotman. Poincaré-Hopf theorems on stratified spaces, 2006. Université de Provence.

[Mac74] R. MacPherson. Chern classes for singular algebraic varieties. *Annals of Mathematics*, 100:423–432, 1974.

[Mac84] R. MacPherson. Global questions in the topology of singular spaces. In *Proceedings of the International Congress of Mathematicians*, pages 213–235, Warszawa, 1984. August 16-24 1983.

[Mat70] J. Mather. Notes on topological stability. Harvard University, 1970.

[Mat73] J. Mather. Stratifications and mappings. *Dynamical Systems*, pages 195–232, 1973.

[Sch65a] M.-H. Schwartz. Classes caractéristiques définies par une stratification d'une variété complexe. *Comptes Rendus de l'Académie des Sciences*, 260:3262–3264 and 3535–3537, 1965.

[Sch65b] M.-H. Schwartz. Classes obstructrices d'un sous-ensemble analytique complexe d'une variété analytique complexe. Publication de l'Université de Lille, 1965.

[Sch86] M.-H. Schwartz. Une généralisation du théorème de Hopf sur les champs sortants. *Comptes Rendus de l'Académie des Sciences*, 303(7):307–309, 1986.

[Sch91] M.-H. Schwartz. *Champs radiaux sur une stratification analytique*. Travaux en cours. Hermann, 1991.

[Sch92] M.-H. Schwartz. Un problème d'homologie dans l'espace des vecteurs tangents à une stratification. *Comptes Rendus de l'Académie des Sciences*, 314:833–836, 1992.

[Sim95] S. Simon. Champs de vecteurs totalement radiaux sur une structure de Thom-Mather. *Annales de l'Institut Fourier*, 45(5):1423–1447, 1995.

[Sul71] D. Sullivan. Combinatorial invariants of analytic spaces. In *Liverpool Singularities Symposium*, volume 192 of *Lecture notes in mathematics*, pages 165–168. Springer-Verlag, 1971.

[Tho69] R. Thom. Ensembles et morphismes stratifiés. *Bulletin of the American Mathematical Society*, 75:240–284, 1969.

[Ver72] A. Verona. Homological properties of abstract prestratifications. *Revue Roumaine de Mathématiques pures et appliquées*, XVII(7):1109–1121, 1972.

[Ver84] A. Verona. *Stratified mappings–Structure and triangulability*, volume 1102 of *Lecture notes in mathematics*. Springer-Verlag Berlin, 1984.